COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 18 (2) (1998) 281–292

ON THE RELATION BETWEEN TWO MINOR-MONOTONE GRAPH PARAMETERS

R. PENDAVINGH

Received June 13, 1997

We prove that for each graph $\mu(G) \le \lambda(G)+2$, where μ and λ are minor-monotone graph invariants introduced by Colin de Verdière [3] and van der Holst, Laurent, and Schrijver [5]. It is also shown that a graph G exists with $\mu(G) < \lambda(G)$. The graphs G with maximal planar complement and $\mu(G) = |V(G)|-4$, characterised by Kotlov, Lovász, and Vempala, are shown to be forbidden minors for $\{H \mid \mu(H) < |V(G)|-4\}$.

1. Introduction

Given a graph $G=(V,E)$ without loops or multiple edges, define $\mathcal{O}_G$ as the collection of real-valued symmetric $V \times V$ matrices $M=(m_{ij})$ satisfying

1. if $ij \in E$, then $m_{ij} < 0$, and
2. if $ij \notin E$ and $i \ne j$, then $m_{ij} = 0$.

There is no restriction on the diagonal entries. The elements of $\mathcal{O}_G$ are sometimes called *discrete Schrödinger operators*.

A matrix $M \in \mathcal{O}_G$ satisfies the *Strong Arnold Hypothesis*, SAH for short, if there is no nonzero symmetric matrix $X=(x_{ij})$ such that $MX=0$, and such that $x_{ij}=0$ whenever $i=j$ or $ij \in E$.

By $\lambda_i(M)$ we denote the i-th smallest eigenvalue of M. When G is connected and $M \in \mathcal{O}_G$, the Perron-Frobenius Theorem implies that for any eigenvector z of M:

$z>0$ or $z<0 \iff z$ belongs to the smallest eigenvalue of M.

Hence, the multiplicity of $\lambda_1(M)$ is 1. The parameter $\mu(G)$ is defined as

the largest corank of any matrix $M \in \mathcal{O}_G$ with exactly one negative eigenvalue, satisfying the SAH.

This definition is due to Colin de Verdière [3]. Equivalently, $\mu(G)$ is the maximum multiplicity of $\lambda_2(M)$, where M ranges over all $M \in \mathcal{O}_G$ satisfying (a more general formulation of) the SAH. See [7] for a survey of results concerning μ.

Mathematics Subject Classification (1991): 05C

If $G=(V,E)$ is a graph and $S\subseteq V$, let $G[S]$ denote the subgraph of G induced by S. Given $x\in\mathbb{R}^V$, the *support* of x is $\text{supp}(x):=\{v\in V\mid x_v\neq 0\}$. Furthermore, $\text{supp}_+(x):=\{v\in V\mid x_v>0\}$ is the postive support, and $\text{supp}_-(x):=\{v\in V\mid x_v<0\}$ is the negative support of a vector x.

A linear subspace $L\subseteq\mathbb{R}^V$ is said to be a *valid representation of* G when for each nonzero $x\in L$, one has

1. $\text{supp}_+(x)\neq\emptyset$, and
2. $G[\text{supp}_+(x)]$ is connected.

Van der Holst, Laurent, and Schrijver [5] defined $\lambda(G)$ as

$$\lambda(G):=\max\{\dim(L)\mid L \text{ is a valid representation of } G\}.$$

When $G=(V,E)$ is a graph and $S\subseteq V$, we denote the set of neighbors of S in G by $N_G(S)$, i.e. $N_G(S):=\{v\in V\setminus S\mid \exists w\in S, vw\in E\}$. When $M\in\mathcal{O}_G$, M_S denotes the restriction of M to the rows and columns indexed by S. Given a vector $x\in\mathbb{R}^V$, $x_S\in\mathbb{R}^S$ denotes the restriction of x to the positions with indices in S. By extension with zeros a vector $x\in R^S$ may still be regarded as an element of R^V, and as such vectors restricted to different subsets of V may be added.

A graph H is a *subgraph* of a graph G, denoted $H\subseteq G$, if H can be obtained by removing vertices and deleting edges from G. When H can be obtained from a subgraph of G by contracting edges, H is a *minor* of G, which we denote by $H\leq G$.

Both λ and μ are *minor-monotone*, i.e.

if H is a minor of G then $\lambda(H)\leq\lambda(G)$ and $\mu(H)\leq\mu(G)$.

The short proof of the minor-monotony of λ appears in [5]. The SAH plays an essential role in proving that μ is minor-monotone. Colin de Verdière's original proof can be found in [3].

A graph $G=(V,E)$ is a *clique sum* of $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ if $V=V_1\cup V_2, E=E_1\cup E_2$ and $V_1\cap V_2$ induces a clique in both G_1 and G_2. If G is a clique sum of G_1 and G_2, then $\lambda(G)\geq\lambda(G_i)$ and $\mu(G)\geq\mu(G_i)$ as G_1 and G_2 are both subgraphs of G. By work of Van der Holst, Laurent, and Schrijver [5], we know that

$$\lambda(G)=\max\{\lambda(G_1),\lambda(G_2)\}.$$

Van der Holst, Lovász, and Schrijver [7] showed that either

1. $\mu(G)\leq\max\{\mu(G_1),\mu(G_2)\}$, or
2. $\mu(G)=\max\{\mu(G_1),\mu(G_2)\}+1$, and $\mu(G_1)=\mu(G_2)$.

It follows inductively that if G is a clique sum of more than two graphs $G_1,\ldots,G_k$, then $\mu(G)\leq\max\{\mu(G_1),\ldots,\mu(G_k)\}+1$.

A $Y\Delta$-operation on G is removing a vertex v of degree 3 from G, and then adding a triangle on the former neighbors of v. The converse is called a ΔY-operation. Bacher and Colin de Verdière show [1]: if H is obtained from G by a $Y\Delta$-operation, then $\mu(H)\leq\mu(G)$. Also, if G is obtained from H by a ΔY-operation, G is a subgraph of a clique sum of H and K_4. Hence, if $\mu(H)\geq 4>\mu(K_4)$, we have $\mu(G)=\mu(H)$.

Given any set of graphs $\mathcal{C}$ closed under taking minors, define the *forbidden minors* for $\mathcal{C}$ as

$$F(\mathcal{C}) := \{G \mid G \notin \mathcal{C}, \forall H < G \; H \in \mathcal{C}\}.$$

Such a set of forbidden minors is finite, by the Robertson-Seymour graph minor Theorem. Clearly, membership of $\mathcal{C}$ can be characterized by

$$H \in \mathcal{C} \Leftrightarrow H \text{ has no element of } F(\mathcal{C}) \text{ as a minor.}$$

Since μ is minor-monotone, the set $\{G \mid \mu(G) < k\}$ is closed under taking minors, for any k. The following forbidden minor characterizations of $\mu < k$ are known:

1. $\mu(G) < 1 \iff G$ has no $\overline{K_2}$-minor,
2. $\mu(G) < 2 \iff G$ has no K_3- or $K_{1,3}$-minor,
3. $\mu(G) < 3 \iff G$ has no K_4- or $K_{2,3}$-minor,
4. $\mu(G) < 4 \iff G$ has no K_5- or $K_{3,3}$-minor,
5. $\mu(G) < 5 \iff G$ has no minor in the Petersen family.

In each of these statements, the '$\Longrightarrow$' part is relatively easy to verify: it suffices to compute μ for the graphs mentioned on the right. All but the last of these results are due to Colin de Verdière [3]. For the characterisation of $\mu < 4$ he used Kuratowski's Theorem, that a graph without K_5- or $K_{3,3}$-minor is planar.

The characterisation of $\mu < 5$ is due to Lovász and Schrijver [9]. They show that so-called 'flat' graphs have $\mu < 5$. By a Theorem of Robertson, Seymour, and Thomas [10], a graph without a minor in the Petersen family is a flat graph. The Petersen family is the set of 7 graphs that can be obtained from K_6 by any series of ΔY- and $Y\Delta$-operations. The Petersen graph is in the Petersen family.

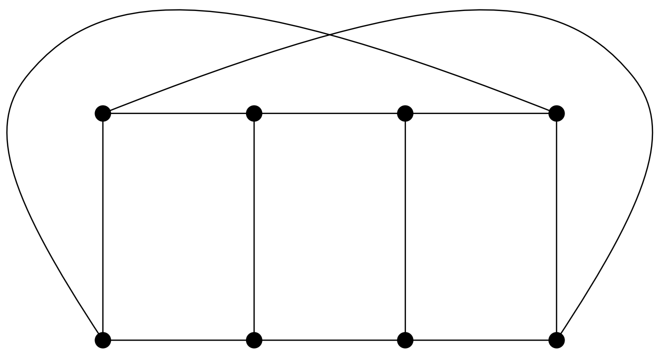

Figure 1. The graph V_8

The corresponding results on λ are [5]:

1. $\lambda(G) < 1 \iff G$ has no $\overline{K_2}$-minor,
2. $\lambda(G) < 2 \iff G$ has no K_3-minor,
3. $\lambda(G) < 3 \iff G$ has no K_4-minor,

4. $\lambda(G) < 4 \iff G$ has no K_5- or V_8-minor.

The graph V_8 is shown in Figure 1.

It was speculated in [9] that the following might be true:

a graph G satisfies $\lambda(G) \le t$ if and only if G is a subgraph of a clique sum of graphs H with $\mu(H) \le t$.

The 'only if' part implies that for any graph G, we have $\mu(G) \le \lambda(G) + 1$. The 'if' part implies that $\mu(G) \ge \lambda(G)$ for all graphs G.

In the next section it is shown that for any graph G, we have $\mu(G) \le \lambda(G) + 2$. In section 4, we find a graph G with $\mu(G) < \lambda(G)$.

2. A relation between μ and λ

The starting point of the present investigation is the following lemma. Its proof appears in [7].

Lemma 1. (van der Holst) *Let G be a connected graph and let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue. Let $x \in \ker M$ be such that $G[\mathrm{supp}_+(x)]$ is disconnected. Then there is no edge between $\mathrm{supp}_+(x)$ and $\mathrm{supp}_-(x)$, and for each component C of $G[\mathrm{supp}(x)]$, $N_G(C) = N_G(\mathrm{supp}(x))$.* ∎

Given a graph $G = (V, E)$, call a vector $x \in \mathbb{R}^V$ *broken* when $G[\mathrm{supp}(x)]$ has at least 3 components. Observe that for any connected graph G, if $M \in \mathcal{O}_G$ has exactly one negative eigenvalue, and x is a nonzero vector in $\ker(M)$, then

1. $\mathrm{supp}_+(x) \neq \emptyset$, and
2. if x is not broken, then $G[\mathrm{supp}_+(x)]$ is connected.

(1. holds as x is orthogonal to an eigenvector $z > 0$ of M corresponding to $\lambda_1(M)$, and 2. is a consequence of Lemma 1.)

So each subspace of $\ker(M)$ avoiding broken vectors is a valid representation of G.

When $M \in \mathcal{O}_G$ has exactly one negative eigenvalue with corresponding eigenvector z, then Rayleigh's Theorem implies

$$x \perp z \text{ and } x^T M x = 0 \iff x \in \ker(M).$$

Using this fact van der Holst, Lovász, and Schrijver [6, 7] show:

Lemma 2. *Let G be a connected graph and $M \in \mathcal{O}_G$ with exactly one negative eigenvalue. Let $S \subseteq V(G)$, and $C_1, \dots, C_k$, the components of $G - S$. Then $\lambda_1(M_{C_i}) < 0$ implies $\lambda_1(M_{C_j}) > 0$ for all $j \neq i$.* ∎

With the techniques that were used to prove Lemma 1 and Lemma 2, it is possible to show the following:

Lemma 3. *Let G be a connected graph and let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue. Let $x \in \ker(M)$, and set*

$$D := \{y \in \ker(M) \mid \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\}.$$

If $G[\operatorname{supp}(x)]$ is disconnected, it has exactly $\dim(D)+1$ connected components. If in addition M satisfies the SAH, then $\dim(D) \leq 2$.

Proof. Let $G[\operatorname{supp}(x)]$ be disconnected, and let $C_1, \ldots, C_k$ be its connected components. $Mx = 0$ implies $M_{C_i} x_{C_i} = 0$, and hence $\lambda_1(M_{C_i}) \leq 0$ for each $i = 1, \ldots, k$.

By Lemma 2, $\lambda_1(M_{C_i}) < 0$ would imply $\lambda_1(M_{C_j}) > 0$ for all $j \neq i$, and thus we have $\lambda_1(M_{C_i}) = 0$ for each $i = 1, \ldots, k$. By the Perron-Frobenius Theorem applied to each M_{C_i} we have either $x_{C_i} < 0$ or $x_{C_i} > 0$. Hence if $y \in D$ then there exists $\alpha_i \in \mathbb{R}$ such that $y_{C_i} = \alpha_i x_{C_i}$, since $M_{C_i} y_{C_i} = 0$. Let z be an eigenvector corresponding to the smallest eigenvalue of M. If $y \in \ker(M)$ then $z^T y = 0$, hence D is contained in

$$D' := \{y \in \mathbb{R}^V \mid y = \sum \alpha_i x_{C_i}, \alpha_i \in \mathbb{R}, z^T y = 0\}.$$

On the other hand, suppose $y \in D'$. Then, since $y^T M y = \sum \alpha_i^2 x_{C_i}^T M_{C_i} x_{C_i} = 0$ and $z^T y = 0$ we have $y \in \ker(M)$ by Rayleigh's Theorem, and certainly $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$, so $D = D'$. Since $z^T y = 0 \Leftrightarrow \sum \alpha_i (z_{C_i}^T x_{C_i}) = 0$, and $z_{C_i}^T x_{C_i} \neq 0$ for all i, it follows that $\dim(D) = \dim(D') = k-1$.

Now assume M satisfies the SAH, and suppose for a contradiction that $G[\operatorname{supp}(x)]$ has more than 3 components. Clearly there exist $s, t \in D$ such that $\operatorname{supp}(s) = C_1 \cup C_2$ and $\operatorname{supp}(t) = C_3 \cup C_4$. But then $X = ts^T + st^T$ is a symmetric matrix such that $MX = 0$ and

$x_{ij} \neq 0 \Rightarrow (i \in C_1 \cup C_2, j \in C_3 \cup C_4$ or vice versa$) \Rightarrow ij \notin E(G)$,
contradicting the SAH. ∎

Theorem 1. *For all connected graphs G, $\mu(G) \leq \lambda(G) + 2$.*

Proof. Let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue and satisfy the SAH, with $\operatorname{corank}(M) = \mu(G)$. By Lemma 3, all broken vectors in $\ker(M)$ are contained in finitely many 2-dimensional subspaces of $\ker(M)$. Then there exists a subspace $L \subseteq \ker(M)$ of dimension $\operatorname{corank}(M) - 2$ that has no nonzero vector in common with any of these subspaces. But then L is a valid representation of G, and $\lambda(G) \geq \dim(L) = \mu(G) - 2$. ∎

The question remains whether this bound is sharp. We do not know any graph G having $\mu(G) = \lambda(G) + 2$, so it may be true that $\mu(G) \leq \lambda(G) + 1$ for all G. Indeed, the latter bound holds for graphs with $\mu(G) \leq 5$, as one verifies knowing the forbidden minor characterizations of $\{G \mid \mu(G) < k\}$ for $k = 1, \ldots 5$. Also, $\mu(K_{k,l}) \leq \min\{k, l\} + 1 = \lambda(K_{k,l}) + 1$ [7]. It is certainly not true that $\mu(G) \leq \lambda(G)$ for all G. For example, $\mu(K_{1,3}) = 2 = \lambda(K_{1,3}) + 1$ and $\mu(P) = 5 = \lambda(P) + 1$ for the Petersen graph P.

3. 3-regular graphs and graphs on a surface

For 3-regular and claw-free graphs, we obtain somewhat better bounds than the $\mu \le \lambda + 2$ of Theorem 1.

Lemma 4. *Let G be a connected graph and let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue. Suppose there is an edge $e = uv \in E(G)$ such that*

$$e \cap \text{supp}(x) \neq \emptyset \text{ for all broken } x \in \ker(M).$$

Then $\text{corank}(M) \le \lambda(G - e) + 1$.

Proof. We show that the space $L := \{y \in \ker(M) \mid y_u + y_v = 0\}$ is a valid representation of $G - e$. On the contrary, suppose that some $y \in L$ is broken. Since for each component C_i of $G[\text{supp}(y)]$ either $y_{C_i} > 0$ or $y_{C_i} < 0$, and by assumption $e \cap \text{supp}(y) \neq \emptyset$, e has an endpoint in two different components of $G[\text{supp}(y)]$, a contradiction with Lemma 1. So y is not broken and hence $G[\text{supp}_+(y)]$ is connected. Because $y_u + y_v = 0$, e does not have both endpoints in $\text{supp}_+(y)$. So even $(G - e)[\text{supp}_+(y)]$ is connected. Hence, L is a valid representation of $G - e$, and $\lambda(G - e) \ge \dim(L) \ge \text{corank}(M) - 1$. ∎

Lemma 5. *Let G be a connected, 3-regular graph and let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue. If $x \in \ker(M)$ is a broken vector, $G - \text{supp}(x)$ has no edges.*

Proof. By Lemma 1 we have $N_G(C) = N_G(\text{supp}(x))$ for each component C of $G[\text{supp}(x)]$. Since G is 3-regular, a vertex $v \notin \text{supp}(x)$ has either $N_G(v) \subseteq \text{supp}(x)$ or $N_G(v) \cap \text{supp}(x) = \emptyset$. By the connectedness of G, if $V \setminus \text{supp}(x) \setminus N_G(\text{supp}(x))$ is nonempty it is connected to $N_G(\text{supp}(x))$, but then there exists a vertex $v \in N_G(\text{supp}(x))$ that is connected to $V \setminus \text{supp}(x) \setminus N_G(\text{supp}(x))$, a contradiction. Hence $V \setminus \text{supp}(x) = N_G(\text{supp}(x))$ and $N_G(v) \subseteq \text{supp}(x)$ for all $v \notin \text{supp}(x)$. ∎

Theorem 2. *For any connected 3-regular graph G, $\mu(G) \le \lambda(G) + 1$.*

Proof. Let $M \in \mathcal{O}_G$ have exactly one eigenvalue and $\text{corank}(M) = \mu(G)$. Let $e \in E(G)$ be any edge. By Lemma 5, $e \cap \text{supp}(x) \neq \emptyset$ for all broken $x \in \ker(M)$. By Lemma 4, $\mu(G) = \text{corank}(M) \le \lambda(G - e) + 1 \le \lambda(G) + 1$. ∎

For example, the Petersen graph P has $\mu(P) = 5$ and $\lambda(P) = 4$, and when an arbitrary edge is removed from P the result is a subdivision of V_8 (see figure 1). The above proof yields a construction of a 4-dimensional valid representation of V_8, given a matrix $M \in \mathcal{O}_P$ with exactly one eigenvalue and $\text{corank}(M) = 5$. It is not necessary that the SAH holds for such a matrix M.

Theorem 3. *If G is a connected claw-free graph, then $\mu(G) \le \lambda(G)$.*

Proof. Suppose $M \in \mathcal{O}_G$ with exactly one eigenvalue and $\text{corank}(M) = \mu(G)$. Suppose $x \in \ker(M)$ is broken. As G is connected, there exists $v \in N_G(\text{supp}(x))$. Then, by Lemma 1, v has neighbors in each of at least 3 components of $\text{supp}(x)$.

This is a contradiction with the assumption that G is claw-free. Hence $\ker(M)$ is a valid representation of G of dimension $\mu(G)$. ∎

Theorem 4. *Given any surface S,*

$$\max\{\mu(H) \mid H \text{ embeds in } S\} \leq \max\{\lambda(H) \mid H \text{ embeds in } S\}.$$

Proof. Let G attain the maximum in $\max\{\mu(H) \mid H$ embeds in $S\}$. We will construct a clawfree graph G' that has G as a minor and is embedded in S. By minor-monotony and Theorem 3 we then have $\mu(G) \leq \mu(G') \leq \lambda(G')$, and the Theorem will follow.

We may assume that all vertices of G have degree at least 3. To obtain G' from G, first split vertices such that embeddability in S is preserved, until each vertex has degree 3. Next, replace each edge by a path of length 2. Finally, add edges connecting each pair of neighbors of a vertex of degree 3. The resulting graph G' has G as a minor by construction, is claw-free and is embedded in S. ∎

4. An upper bound on μ, and a counterexample

Theorem 5. *If $G=(V,E)$ is any connected graph, then either*

$$|E| \geq \mu(G)(\mu(G)+1)/2 \text{ or } G = K_{3,3}.$$

Proof. Suppose there exists some $G \neq K_{3,3}$ with $|E| < \mu(G)(\mu(G)+1)/2$. Then $\mu(G) > 4$, and we may assume that G is triangle-free, as applying ΔY operations does not violate the condition that $|E| < \mu(G)(\mu(G)+1)/2$.

Let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue and satisfy the SAH, with $\operatorname{corank}(M) = \mu(G)$.

Suppose that some diagonal entry of M is nonzero, say $m_{11} \neq 0$. Let $F \subseteq E$ be the edges of a spanning tree of G. The linear space of matrices

$$\mathcal{X} := \{X \mid X \text{ symmetric and } MX = 0\}$$

has dimension $\dim(\mathcal{X}) = \mu(G)(\mu(G)+1)/2$. Consider the subspace

$$\mathcal{X}' := \{X \in \mathcal{X} \mid \forall ij \in E \setminus F\ x_{ij} = 0, \forall i \neq 1\ x_{ii} = 0\}.$$

Let $X \in \mathcal{X}'$, and set $F' := \{ij \in E \mid x_{ij} \neq 0\}$. As $F' \subseteq F$, the graph $G' = (V, F')$ is a forest. If $F' \neq \emptyset$, then there exists some vertex $i \neq 1$ of degree 1 in G'. This contradicts the fact that $x_{ii} = 0$ and $m_i^T x_i = 0$. Hence, $x_{ij} = 0$ for all $ij \in E$. Similarly, $m_1^T x_1 = 0$ and $m_{11} \neq 0$ imply $x_{11} = 0$. So also $x_{ii} = 0$ for all $i \in V$.

Since the SAH holds for M we have $X=0$. It follows that $\mathcal{X}'$ contains no nonzero elements. From this, and the definition of $\mathcal{X}'$ it follows that

$$0 = \dim(\mathcal{X}') \geq \dim(\mathcal{X}) - |E \setminus F| - (|V| - 1)$$
$$= \mu(G)(\mu(G) + 1)/2 - |E| > 0,$$

a contradiction. So $m_{ii}=0$ for all $i \in V$.

Let $z > 0$ be an eigenvector belonging to the smallest eigenvalue of M. If $ij \notin E$, then the vector u defined by $u_i = z_j$, $u_j = -z_i$ and $u_k = 0$ when $k \neq i,j$ has $z^T u = 0$ and $u^T M u = 0$. Hence, $u \in \ker(M)$ by Rayleigh's Theorem. By Lemma 1 applied to u, it follows that $N(i) = N(j)$. Hence, G is complete multipartite. If C is a coclique of G, then we similarly find an $x \in \ker(M)$ with $\mathrm{supp}(x) = C$. Hence $|C| \leq 3$ by Lemma 3. Since G is also triangle-free, G is an induced subgraph of $K_{3,3}$, contradicting the fact that $\mu(G) > 4$. ∎

For any k, the complete graph K_{k+1} is a forbidden minor for $\{H \mid \mu(H) < k\}$. Also, if $k > 4$ then each graph obtained from K_{k+1} by any series of ΔY- and $Y\Delta$-operations is a forbidden minor, and all these graphs have the same number of edges as K_{k+1}. In general, not every forbidden minor is obtained this way, but we do have:

Corollary 5.1. *If G is a forbidden minor for $\{H \mid \mu(H) < k\}$, then G has at least as many edges as the complete graph K_{k+1}, or $G = K_{3,3}$.* ∎

Kotlov, Lovász, and Vempala [8] characterise the graphs G whose complement $\overline{G}$ is a maximal planar graph and for which $\mu(G) \geq |V(G)| - 4$.

Corollary 5.2. *If $\overline{G}$ is a maximal planar graph and $\mu(G) = |V(G)| - 4$, then G is a forbidden minor for $\{H \mid \mu(H) < |V(G)| - 4\}$.*

Proof. If $\overline{G}$ is a maximal planar graph, then $|E(\overline{G})| = 3|V(G)| - 6$. Hence, $|E(G)| = (|V(G)| - 4)(|V(G)| - 3)/2$. Any proper minor H of G has strictly fewer edges than G, and hence by Theorem 5 we have $\mu(H) < |V(G)| - 4$. ∎

For example, we have $\mu(\overline{I}) = |V(\overline{I})| - 4 = 8$ where I denotes the Icosahedron. So $\overline{I}$ is a forbidden minor for $\{H \mid \mu(H) < 8\}$, as is K_9. One cannot obtain $\overline{I}$ from K_9 by a series of ΔY- and $Y\Delta$-operations (shown by computer).

A *generalised dodecagon* of order (1,2) or $GD(1,2)$ is a graph G with the following properties:

1. G is 3-regular,
2. G has diameter 6,
3. for each vertex $v \in V$ and each $i = 1, \ldots, 5$: if $u \in V$ is at distance i from v, then u has 2 neighbors at distance $i+1$ from v and 1 neighbor at distance $i-1$ from v, and

4. for each vertex $v \in V$: if $u \in V$ has distance 6 from v, then it has 3 neighbors at distance 5 from v.

In other words, a $GD(1,2)$ is a distance-regular graph with intersection array $\{3,2,2,2,2,2;1,1,1,1,1,3\}$. A $GD(1,2)$ has 126 vertices, and 189 edges. There exists a unique generalised dodecagon of order $(1,2)$ [2], known as Tutte's 12-Cage. We will denote this graph by T.

Theorem 6. $\mu(T) \leq 18 < 20 \leq \lambda(T)$.

Proof. By Theorem 5, $\mu(T) \leq 18$ as $|E(T)| = 189$ and $T \neq K_{3,3}$. The second-largest eigenvalue θ of the adjacency matrix A of T has multiplicity 21 [2, p. 416]. Hence $M := \theta I - A \in \mathcal{O}_T$ has exactly one negative eigenvalue, and $\text{corank}(M) = 21$. As T is 3-regular, $\lambda(T) \geq 20$ by the proof of Theorem 2. ∎

5. Lower bounds for λ

The following theorem is due to Kotlov, Lovász, and Vempala [8]:

Theorem 7. *For every graph G,*

1. *if $\overline{G}$ is a disjoint union of paths, then $\mu(G) \geq |V(G)| - 3$,*
2. *if $\overline{G}$ is outerplanar, then $\mu(G) \geq |V(G)| - 4$,*
3. *if $\overline{G}$ is planar, then $\mu(G) \geq |V(G)| - 5$.* ∎

These bounds on μ can be extended to λ as follows:

Theorem 8. *For every graph G,*

1. *if $\overline{G}$ is a disjoint union of paths, then $\lambda(G) \geq |V(G)| - 3$,*
2. *if $\overline{G}$ is outerplanar, then $\lambda(G) \geq |V(G)| - 4$,*
3. *if $\overline{G}$ is planar and $K_{2,2,2} \not\subseteq \overline{G}$, then $\lambda(G) \geq |V(G)| - 5$.*

Proof. 1. When $\overline{G}$ is a disjoint union of paths, $\mu(G) \geq |V(G)| - 3$. Also, G is claw-free as $\overline{G}$ has no triangles. By Theorem 3, $\mu(G) \leq \lambda(G)$.

2. Observe that if $\overline{H}$ is a subgraph of $\overline{G}$, then $|V(H)| - \lambda(H) \leq |V(G)| - \lambda(G)$. We may therefore assume that $\overline{G}$ is maximally outerplanar. So $\overline{G}$ is 2-connected, contains no $K_{2,3}$- or K_4-minor, and has $\mu(G) \geq |V(G)| - 4$.

Let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue and $\text{corank}(M) = \mu(G)$. Suppose $x \in \ker(M)$ is broken. Let C_1, C_2, C_3 be components of $G[\text{supp}(x)]$, indexed such that $|C_1| \geq |C_2| \geq |C_3|$.

Since $K_{|C_1|,|C_2|,|C_3|}$ is a subgraph of $\overline{G}[\text{supp}(x)]$, and $K_{2,3}$ is not a subgraph of $\overline{G}$, either $(|C_1|,|C_2|,|C_3|) = (1,1,1)$ or $(|C_1|,|C_2|,|C_3|) = (2,1,1)$. Similarly,

$V\setminus \mathrm{supp}(x) = N_G(\mathrm{supp}(x))$ as K_4 is not a subgraph of $\overline{G}$. Hence, if C_i is a singleton this implies that C_i is not connected to $V\setminus\mathrm{supp}(x)$ in $\overline{G}$. As a maximally outerplanar graph is 2-connected, $(|C_1|,|C_2|,|C_3|) = (1,1,1)$ entails $\mathrm{supp}(x) = V$, and we are done. When $(|C_1|,|C_2|,|C_3|)=(2,1,1)$, $v\notin \mathrm{supp}(x)$ must be connected to C_1 by 2 vertex-disjoint paths in $\overline{G}$. This implies $\overline{G}\geq K_{2,3}$, a contradiction. So $\mathrm{supp}(x)=V$ and we are done.

Since $\ker(M)$ contains no broken vectors, it is a valid representation of G.

3. We may assume that $\overline{G}$ is maximally planar, no vertex has degree 4 in $\overline{G}$, and $K_{2,2,2}\not\subseteq\overline{G}$, by the following argument. If a graph is not maximally planar and does not contain a $K_{2,2,2}$-subgraph, it is always possible to add an edge, keeping planar and not introducing a $K_{2,2,2}$-subgraph. Furthermore, when each triangular face of a maximally planar graph is subdivided by a vertex, the resulting graph contains no vertex of degree 4. Since $\overline{G}$ is maximally planar, $\overline{G}$ is 3-connected and contains no $K_{3,3}$- or K_5-minor.

Let $M \in \mathcal{O}_G$ have exactly one negative eigenvalue and $\mathrm{corank}(M) = \mu(G)$, and let M satisfy the SAH. Suppose $x\in\ker(M)$ is broken. Let C_1,C_2,C_3 be the components of $G[\mathrm{supp}(x)]$, indexed such that $|C_1|\geq|C_2|\geq|C_3|$.

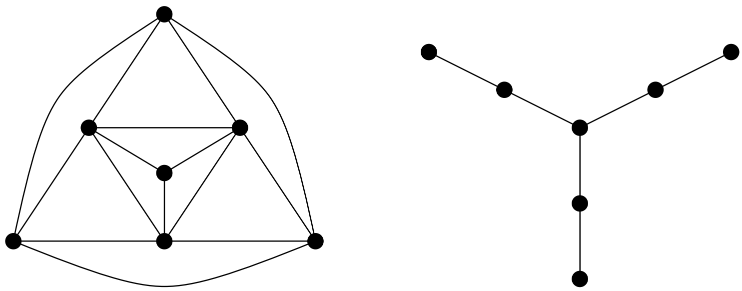

Figure 2. $\overline{G}$ (left) is max. planar, and G (right) is a tree.

Since $K_{3,3}\not\subseteq\overline{G}[\mathrm{supp}(x)]$ we have that if $|C_1|\geq 3$ then $|C_1| = |C_2| = 1$. Hence there are the following cases:

Case 1. $(|C_1|,|C_2|,|C_3|) = (1,1,1)$: $v\in N_G(\mathrm{supp}(x))$ and 3-connectivity imply $|V\setminus\mathrm{supp}(x)\setminus N_G(\mathrm{supp}(x))|\geq 3$,as v must be 3-connected in $\overline{G}$ to the vertex in C_3. Hence $K_{3,3}\subseteq\overline{G}-N_G(\mathrm{supp}(x))$, a contradiction. So $V=C_1\cup C_2\cup C_3$, done.

Case 2. $(|C_1|,|C_2|,|C_3|) = (2,1,1)$: $v\in N_G(\mathrm{supp}(x))$ and 3-connectivity imply $|V\setminus\mathrm{supp}(x)\setminus N_G(\mathrm{supp}(x))|\geq 1$. As $K_5\not\subseteq\overline{G}-N_G(\mathrm{supp}(x))$, we know $|V\setminus\mathrm{supp}(x)\setminus N_G(\mathrm{supp}(x))| = 1$. So $\overline{G}-N_G(\mathrm{supp}(x)) = K_{2,1,1,1}$. Since $\overline{G}$ is maximally planar, $N_G(\mathrm{supp}(x))=\emptyset$. So $\overline{G}=K_{2,1,1,1}$ and hence $\lambda(G)\geq|V(G)|-5$.

Case 3. $(|C_1|,|C_2|,|C_3|)=(k,1,1)$, $k>2$: Now, $V\setminus\mathrm{supp}(x)\setminus N_G(\mathrm{supp}(x))=\emptyset$ since otherwise $K_{3,3}\subseteq\overline{G}-N_G(\mathrm{supp}(x))$. Also, $N_G(\mathrm{supp}(x))=\emptyset$, since $v\in N_G(\mathrm{supp}(x))$

would be 3-connected to C_1, which implies $\overline{G} \geq K_{3,3}$. So $V = C_1 \cup C_2 \cup C_3$, and as $\overline{G}$ is maximally planar, $\overline{G}[C_1]$ is a cycle. Hence, any $v \in C_1$ has degree 4 in $\overline{G}$, a contradiction.

Case 4. $(|C_1|, |C_2|, |C_3|) = (2,2,1)$: then $V \setminus \text{supp}(x) \setminus N_G(\text{supp}(x)) = \emptyset$. Hence, the unique vertex $v \in C_3$ has degree 4 in $\overline{G}$, contradiction.

Case 5. $(|C_1|, |C_2|, |C_3|) = (2,2,2)$: then $K_{2,2,2} \subseteq \overline{G}$, contradicting our assumption.

Since $\ker(M)$ contains no broken vectors, it is a valid representation of G. ∎

The graph G as shown in figure 2 has a planar complement, but $K_{2,2,2} \subseteq \overline{G}$ and $\lambda(G) = 1 \not\geq 2 = |V(G)| - 5$.

References

[1] R. Bacher, Y. Colin de Verdière: Multiplicités des valeurs propres et transformations étoile-triangle des graphes, *Bulletin de la Société Mathématique de France*, **123** (1995), 101–117.

[2] A. E. Brouwer, A. M. Cohen, A. Neumaier: *Distance-Regular Graphs*, Springer Verlag, Ergebnisse der Mathematik und ihrer Grensgebiete 3. Folge, Band 18, 1989.

[3] Y. Colin de Verdière: On a new graph invariant and a criterion for planarity in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary mathematics, American Mathematical Society, Providence, Rhode Island, 1993, 137–147.

[4] H. van der Holst: Topological and Spectral Graph Characterisations, Ph.D. Thesis, University of Amsterdam (1996).

[5] H. van der Holst, M. Laurent, and A. Schrijver: On a minor-monotone graph invariant, *J. Combin. Theory Ser. B*, **65** (1995), 291–304.

[6] H. van der Holst, L. Lovász, A. Schrijver: On the invariance of Colin de Verdière's graph parameter under clique sums, *Linear Algebra and its Applications*, **226** (1995), 509–517.

[7] H. van der Holst, L. Lovász, A. Schrijver: The Colin de Verdière graph parameter, to appear in the proceedings of the 16th British Combinatorial Conference.

[8] A. Kotlov, L. Lovász, S. Vempala: The Colin de Verdière number and sphere representations of a graph, preprint, 1996.

[9] L. Lovász and A. Schrijver: The Colin de Verdière number of linklessly embeddable graphs, *Proceedings of the American Mathematical Society*, to appear.

[10] N. Roberytson, P. Seymour, and R. Thomas: Sachs' Linkless Embedding Conjecture, *Combin. Theory Ser. B*, **64** (1995), 185–227.

R. Pendavingh

Faculteit der Wiskunde,
Informatica, Natuurkunde en Sterrenkunde
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam